ON SEISMIC RAYS AND WAVES*

(Part One)

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ABSTRACT

THE EQUATIONS of motion of an elastico-viscous medium in which the material constants vary with position are deduced. These can be put into the form of a wave equation only when the gradients of the constants are small. By the method of Sommerfeld and Runge these equations are compared with the equation of the characteristic function, whence the condition for the validity of the ray method is obtained. It is similar to De Broglie's criterion in wave mechanics. Expressed in terms of measurable quantities in seismology, the condition is applied to the data recently obtained by Gutenberg for the upper layers of the earth's crust. The equation of the characteristic function is used in deriving the forms of the ray paths for several particular velocity functions, following a method previously used by Epstein.

INTRODUCTION

THE THEORY of various seismic phenomena as now accepted has its basis in classical mechanics and particularly in that branch which deals with small displacements in an elastic medium. As such, the theory is well founded and sufficiently developed; it requires neither new hypothesis nor new principles. The physics in this domain is clear, even though mathematical technique is yet to be developed to give explicit solutions of some particular problems.

Nevertheless, a large part of seismological calculations is derived not directly from the equations of general mechanics, but from the concept of seismic rays, after the manner of geometrical optics. It is important to point out that Fermat's principle in optics is the result of our experience with the waves of visible light, the lengths of which are small as compared with ordinary dimensions, whereas in seismology, and especially in seismic prospecting, the dimensions of the quantities to be determined and the width of the seismic disturbance which we use to determine them are often not too different in order of magnitude. The presumption of Fermat's principle in these cases, therefore, requires justification, both experimentally and theoretically.

Furthermore, from the concept of seismic rays alone in conjunction with the generalized Fermat's principle, all we can expect to find is the geometry of the trajectory along which a seismic disturbance is traveling. The notion of energy, or, more correctly, of the intensity of energy flux, is not inherent in this principle. The distribution of the intensity pertaining to a certain pencil of seismic

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rays actually forms an independent assumption and cannot be deduced from the characteristic equation of the ray theory. It is in this mixed domain, where the principle of stationary time and the principle of energy are simultaneously considered, that paradox sometimes arises and leads to inexplicable difficulties. This is exemplified in the case of the so-called "refraction" phenomenon currently understood in applied seismology.

In Hamilton's method, a wave surface is one which is drawn orthogonal to the rays and which moves, in course of time, in a direction normal to itself with a speed of propagation V. If we designate this surface by S, then S is a function of both the coördinates and time and its motion satisfies a partial differential equation of the first order but of the second degree, which is known as Hamilton's equation of the characteristic function.

On the other hand, without asserting Fermat's principle, the validity of which in seismology we have merely taken for granted, we can start from the equations of motion for a continuous medium. By use of the linear stress-strain relations, the validity and limitation of which are well known, we can arrive at a partial differential equation of the second order but of the first degree. This is not the same as Hamilton's equation, and can be made consistent with it only when the wave lengths are very small. The ray method is in reality a limiting case of the more rigorous theory.

These ideas have been extensively developed in other branches of physics, but they do not seem to have received sufficient attention in seismology. The clarification assumes some significance, because quite frequently we are dealing with wave lengths which are neither too long nor too short. Since the ray method is usually adopted in practice, it is important to find the criterion for its validity and thereby to gain some idea of how far we can draw definite conclusions. Starting from the fundamental equations, we shall make a systematic study of the various phases of the propagation problem of seismic disturbances, both as rays and as waves. We shall begin by examining the foundations of the ray theory. A few examples for finding the ray paths will be worked out by adopting the method of Jacobi in solving the equations of motion in general mechanics.

Equations of Motion

In theoretical seismology, two assumptions,[1] among others, are frequently made: (1) that the medium is perfectly elastic, so that the deformations are independent of time; (2) that the medium is homogeneous, so that the elastic constants are independent of position. We propose to depart slightly from these assumptions and examine the consequences.

Following Butcher[2] and Jeffreys,[3] we assume the medium to be elastico-

[1] B. Gutenberg, *Handbuch der Geophysik*, Bd. 4, S. 1 (1932).

[2] J. G. Butcher, "On Viscous Fluids in Motion," *Proc. Lond. Math. Soc.*, 8: 103–135 (1876).

[3] H. Jeffreys, "On Plasticity and Creep in Solids," *Proc. Roy. Soc. Lond.* A, Vol. 138, pp. 283–297 (1932).

viscous so that the rate of increase of strain may be decomposed into two parts:

$$\frac{\partial e_{ik}}{\partial t} = \frac{\partial e_{ik1}}{\partial t} + \frac{\partial e_{ik2}}{\partial t} \tag{1}$$

of which e_{ik1} obeys Hooke's law as in an elastic solid and $\frac{\partial e_{ik2}}{\partial t}$ is linearly related to the stress as in a viscous fluid. Hence, we obtain, in the usual notations,

$$X_x = \lambda_1\theta_1 + 2\mu_1\frac{\partial u_1}{\partial x}, \qquad X_y = \mu_1\left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right) \tag{2}$$

$$X_x = \lambda_2\frac{\partial \theta_2}{\partial t} + 2\mu_2\frac{\partial^2 u_2}{\partial t\partial x}, \qquad X_y = \mu_2\frac{\partial}{\partial t}\left(\frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial x}\right) \tag{3}$$

and similar expressions for the other components, where the λ's and μ's are the Lamé constants for the two parts of the strain, the θ's are the corresponding dilatations, and u, v, w are the components of the displacement. It follows that the sum of the normal stresses is given by

$$-3p = X_x + Y_y + Z_z = (3\lambda_1 + 2\mu_1)\theta_1 = (3\lambda_2 + 2\mu_2)\frac{\partial \theta_2}{\partial t}$$

On the basis of the experimental evidence that a constant symmetrical stress does not produce an appreciable change of volume which increases indefinitely with time, Jeffreys concludes that λ_2 must be extremely large. If there is no dilatation initially, then $\theta_2 = 0$, $\theta = \theta_1$, and he obtains

$$\begin{aligned} \frac{\partial}{\partial t}X_x + \frac{\mu_1}{\mu_2}X_x &= \lambda_1\frac{\partial \theta_1}{\partial t} + \frac{\mu_1}{\mu_2}k_1\theta_1 + 2\mu_1\frac{\partial^2 u}{\partial t\partial x} \\ \frac{\partial}{\partial t}X_y + \frac{\mu_1}{\mu_2}X_y &= \mu_1\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \end{aligned} \tag{4}$$

and similar expressions for the other components, where u and v are written for $u_1 + u_2$ and $v_1 + v_2$ respectively in accordance with (1), $k_1 = \lambda_1 + \frac{2}{3}\mu_1$ is the bulk modulus and μ_2 is of the dimensions of the coefficient of viscosity.

Limiting our consideration to time variations of the simple harmonic type type (we can of course generalize it, at least in principle, to other types by use of Fourier's theorem), we shall associate all motions with a time factor $e^{-i\omega t}$, ω

being the angular frequency. Then, a time differentiation is equivalent to a multiplication with $-i\omega$, and equations (4) become

$$
\begin{aligned}
X_x &= \lambda\theta + 2\mu \frac{\partial u}{\partial x} & X_y &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
Y_y &= \lambda\theta + 2\mu \frac{\partial v}{\partial y} & Y_z &= \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\
Z_z &= \lambda\theta + 2\mu \frac{\partial w}{\partial z} & Z_x &= \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)
\end{aligned}
\tag{5}
$$

where

$$
\lambda = \frac{k_1\mu_1 - i\omega\lambda_1\mu_2}{\mu_1 - i\omega\mu_2} \qquad \mu = \frac{i\omega\mu_1\mu_2}{i\omega\mu_2 - \mu_1}\ * \tag{6}
$$

Equations (5) hold when the motion is simple harmonic in time and they are of the same forms as those for an elastic medium. From (6) it is seen that, when the frequency is very high, $\lambda \rightarrow \lambda_1$ and $\mu \rightarrow \mu_1$ and the medium behaves like an elastic solid. When the frequency is very low, $\lambda \rightarrow k_1$ and μ becomes very small. The medium, in this case, behaves like a compressible fluid.

So far, we have said nothing about the inhomogeneity of the medium. Let us now consider the equation of motion

$$
\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}
$$

and two similar equations for the components v and w. Substituting the stress components from (5) and taking account of the fact that λ, μ, and ρ are all functions of the coördinates, we obtain, after some simplification,

$$
\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + \mu)\frac{\partial \theta}{\partial x} + \mu\nabla^2 u + \theta\frac{\partial \lambda}{\partial x} + \nabla\mu \cdot \left(\nabla u + \frac{\partial \bar{S}}{\partial x}\right) \tag{7}
$$

where $\bar{S}$ is the displacement vector the components of which are u, v, and w. The expressions for $\frac{\partial^2 v}{\partial t^2}$ and $\frac{\partial^2 w}{\partial t^2}$ are similar. If we add up these three components vectorially, we obtain the simple relation

$$
\rho \frac{\partial^2 \bar{S}}{\partial t^2} = (\lambda + \mu)\nabla\theta + \mu\nabla^2\bar{S} + \theta\nabla\lambda + 2\nabla\mu \cdot \nabla\bar{S} \tag{8}
$$

* We are not too much concerned here with the particular hypothesis of the plastic state of the medium, e.g., be it elastico-viscous in the sense of Maxwell's conception, or visco-elastic in the sense of Voigt's. What we wish to emphasize is merely the signs of the imaginary parts of these quantities.

As in the ordinary case, we may perform the operations divergence and curl on both sides of (8) successively. The results are

$$\rho \frac{\partial^2\theta}{\partial t^2} + \nabla\rho \cdot \frac{\partial^2 \bar{S}}{\partial t^2} = (\lambda + 2\mu)\nabla^2\theta + \nabla\theta \cdot \nabla(\lambda + \mu) + \nabla\mu \cdot \nabla^2\bar{S} + \nabla \cdot (\theta\nabla\lambda) + 2\nabla \cdot (\nabla\mu \cdot \nabla\bar{S}) \tag{9}$$

$$\rho \frac{\partial^2\tilde{\omega}}{\partial t^2} - \nabla\rho \times \frac{\partial^2 \bar{S}}{\partial t^2} = \mu\nabla^2\tilde{\omega} + \nabla\theta \times \nabla(\lambda + \mu) + \nabla^2\bar{S} \times \nabla\mu + \nabla \times (\theta\nabla\lambda) + 2\nabla \times (\nabla\mu \cdot \nabla\bar{S}) \tag{10}$$

where $\tilde{\omega} = \nabla \times \bar{S}$ is the rotation. It is seen from these last two equations that the dilatation θ and the rotation $\tilde{\omega}$ are generally not separable if the parameters λ, μ, and ρ vary with position.

Equations (9) and (10) will be reduced to the familiar equations for the longitudinal and transverse waves if we retain only the first terms on both sides. This is permissible only if the gradients of the parameters $\nabla\lambda$, $\nabla\mu$, and $\nabla\rho$ are very small as compared with the other quantities in the equations. That implies that the changes of the properties of the medium should be so gradual that within distances of the order of magnitude of a wave length the medium may be regarded as homogeneous. This condition is frequently satisfied in seismic phenomena, but not always. When it is, we can write

$$\rho \frac{\partial^2\theta}{\partial t^2} = (\lambda + 2\mu)\nabla^2\theta \tag{11}$$

$$\rho \frac{\partial^2\tilde{\omega}}{\partial t^2} = \mu\nabla^2\tilde{\omega} \tag{12}$$

and the dilatation and distortion are separated. The foregoing equations are of the form of a wave equation

$$\nabla^2\varphi = \frac{1}{V^2}\frac{\partial^2\varphi}{\partial t^2} \tag{13}$$

where

$$V^2 = (\lambda + 2\mu)/\rho \quad \text{or} \quad \mu/\rho$$

as the case may be. It must be emphasized the V is *not a constant* here, but a slowly varying function of position, i.e.,

$$V = V(x, y, z)\,.$$

If the medium is perfectly elastic, V is real and equal to the velocity of propagation of a wave front in the medium. However, as can be seen from (6), V is generally complex. Its physical interpretation must then be sought from other considerations.

Seismic Rays and the Phase Velocity

In seismic interpretations it has been found convenient to use the notions of frequency ν, wave length λ, and phase Θ, which were originally referred to simple harmonic waves of infinite extent. Let us again confine our attention to time variations of the type $e^{-i\omega t}$, where $\omega = 2\pi\nu$. We can then write a continuous function $\varphi = \varphi(x, y, z, t)$ in the form

$$\varphi = \psi e^{-i\omega t} = Ce^{-i\omega t}e^{ik_0 S} \tag{14}$$

where

$$S(x, y, z) = S_1(x, y, z) + iS_2(x, y, z) \tag{15}$$

is a complex function and $C = C_0 e^{ib}$, a complex constant. The constant k_0 which is equal to the ratio of ω to v_0, the latter being a real velocity referred to an arbitrarily chosen ideal medium, is introduced merely for convenience. The real part of φ is given by

$$\text{Re}\,\varphi = C_0 e^{-k_0 S_2} \cos(\omega t - k_0 S_1 - b) = A \cos \Theta \tag{16}$$

Let us identify S_1 with the wave surface which is drawn perpendicular to the rays. The expression given above represents a progressive wave with amplitude $A = C_0 e^{-k_0 S_2}$ and phase

$$\Theta = \omega t - k_0 S_1 - b\,. \tag{17}$$

At a given time, both A and Θ depend on the position. The surfaces $S_1(x, y, z) =$ constant, and $S_2(x, y, z) =$ constant, on which the phase and amplitude have constant values respectively, do not coincide in general.

If we fix our attention to a certain value of the phase, then, as time goes on, this same phase value progresses from one surface to another. We define the *phase velocity* v as that with which a given value of the phase advances in the direction of the normal to this equiphase surface, or, in other words, along the ray. Here, v is a real, physical quantity.

In order that the value of a phase Θ remains unchanged as it advances, it must be such that its change with time is exactly compensated by its change with position. This means that

$$\frac{\partial \Theta}{\partial t} = -v \frac{\partial \Theta}{\partial n}$$

n being the distance measured along the normal or the ray. It is this last directional derivative along the normal which implies the use of Fermat's principle. From (17), we have

$$\frac{\partial \Theta}{\partial t} = \omega\,, \qquad \frac{\partial \Theta}{\partial n} = -k_0 \frac{\partial S_1}{\partial n} = -\frac{\omega}{v_0}\,|\nabla S_1|\,.$$

Hence, we obtain

$$(\nabla S_1)^2 = \left(\frac{\partial S_1}{\partial x}\right)^2 + \left(\frac{\partial S_1}{\partial y}\right)^2 + \left(\frac{\partial S_1}{\partial z}\right)^2 = \left(\frac{v_0}{v}\right)^2 = n^2 \tag{18}$$

where

$$n = v_0/v = k/k_0$$

is the index of refraction referred to an arbitrarily chosen medium. This last equation is known as Hamilton's equation of the characteristic function (also known as the equation of the eikonals). Since S_1, v_0, and v are all real, if we write $dS_1 = v_0 dt$, equation (18) may be written as

$$\left(\frac{\partial t}{\partial x}\right)^2 + \left(\frac{\partial t}{\partial y}\right)^2 + \left(\frac{\partial t}{\partial z}\right)^2 = \frac{1}{v^2} \tag{18a}$$

and t is evidently the time of travel along a ray from any assigned phase surface. In fact,

$$S_1 = v_0 \int dt + \text{constant} = v_0 \int dr/v + \text{constant}\,,$$

the integration being taken along the ray, and we have

$$\operatorname{Re}\varphi = A\cos(\omega t - k_0 S_1 - b) = A\cos\omega(t - \textstyle\int dr/v + \epsilon) \tag{19}$$

where ϵ is an arbitrary phase constant.

If we generalize the definition of v and let it assume also complex values, then (18) is no longer valid as is evident, because both S_1 and v_0 are real. The more general relation

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 = \left(\frac{v_0}{v}\right)^2 \tag{20}$$

has been deduced by Epstein[4] from a variational principle. By the formal sub-

[4] P. S. Epstein, "Geometrical Optics in Absorbing Media," *Proc. Nat. Acad. Sci.*, 16: 37–45 (1930).

stitution $dS = v_0 dt$, we obtain again the equation (18a), but we can no longer interpret t as the time, because S and therefore also t are complex. Equation (20) may be regarded as a generalization of Fermat's principle.

To see how approximate is the relation (20) and how the phase velocity v is related to the velocity of propagation V in (13), we follow the method of Sommerfeld and Runge.[5] Let us identify the function φ in (14) with that in (13) and transform the latter by the substitutions

$$\varphi = e^{-i\omega t}\psi, \qquad k = \omega/V, \qquad n = v_0/V = k/k_0\,.$$

We then obtain

$$\nabla^2\psi + n^2 k_0 \psi = 0 \tag{21}$$

In order that the expression (14) may represent a true motion, ψ must satisfy (21). If, further, we let $\psi = Ce^{ik_0 S}$, we have

$$\nabla\psi = ik_0 C \nabla S e^{ik_0 S}\,,$$

$$\nabla^2\psi = [ik_0 C \nabla^2 S - k_0{}^2 C(\nabla S)^2] e^{ik_0 S}\,.$$

Hence (21) becomes

$$ik_0 \nabla^2 S - k_0{}^2 (\nabla S)^2 + n^2 k_0{}^2 = 0\,. \tag{22}$$

If

$$|\nabla^2 S| << k_0 (\nabla S)^2\,, \tag{23}$$

then,

$$(\nabla S)^2 = n^2$$

or

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 = \left(\frac{v_0}{V}\right)^2 \tag{24}$$

If V or n^2 is real, so is S and therefore $S = S_1$. Comparing (24) with (18), we see at once that $V = v$. Hence, the velocity of propagation is the same as the phase velocity defined above, but with the provision that (23) holds.

We can thus arrive at a more general equation of the ray theory by regarding it as a limiting case of the wave theory. The condition is given by (23). An integral of (20) will give both the surface of equal phase and that of equal amplitude. One consequence of a complex v is that the amplitude cannot be constant, because S being complex, S_2 does not vanish.

[5] A. Sommerfeld und J. Runge, "Anwendung der Vektorrechnung auf die Grundlagen der geometrischen Optik," *Ann. d. Phys.*, 35: 277–298 (1911).

Limitation of the Ray Method

In order that the ray method may be unambiguously applicable, the condition (23) must be satisfied. The question at hand is to transform this into a form expressed in terms of quantities usually measured. We shall only consider the case of a real V. Let the direction cosines of the normal to the phase surface be l, m, n. We have, from (20),

$$\frac{\partial S}{\partial x} = l\frac{\partial S}{\partial n} = l\frac{v_0}{V}$$

$$\frac{\partial^2 S}{\partial x^2} = -l\frac{v_0}{V^2}\frac{\partial V}{\partial x}$$

$$\nabla^2 S = -\frac{v_0}{V^2}\left(l\frac{\partial V}{\partial x} + m\frac{\partial V}{\partial y} + n\frac{\partial V}{\partial z}\right) = -\frac{v_0}{V^2}\frac{\partial V}{\partial n}$$

Hence, condition (23) becomes

$$\frac{\partial V}{\partial n} << \omega \tag{25}$$

For any direction s making an angle θ with n, we have

$$\cos\theta\frac{\partial V}{\partial s} << \omega \tag{25a}$$

If we are dealing with a simple harmonic wave of length λ and period T, we may also express (25) in the form

$$\frac{\cos\theta}{V}\frac{\partial V}{\partial s} << \frac{2\pi}{\lambda} \tag{25b}$$

or

$$\cos\theta\frac{\partial V}{\partial s} << \frac{2\pi}{T} \tag{25c}$$

In the form (25b) this condition has been given by L. de Broglie[6] in quantum mechanics. Since $\cos\theta$ is always less than unity, it will be sufficient if the inequalities hold without this factor. An immediate consequence of (25) is that the ray method is always applicable when the frequency is sufficiently high.

[6] L. de Broglie, "Les Principes de la nouvelle mécanique ondulatoire," *Jour. de Physique et le Radium,* 7(6): 321–337 (1926).

The case of the most interest in seismology is that when V is a function of the single variable z, the depth from the earth's surface. By the method of Wiechert and Bateman, which is based on Fermat's principle, the velocity of seismic waves at different distances from the earth's center has been calculated by several authorities. Arakawa[7] applied De Broglie's condition to the data calculated by Honda and has attempted to find the depth at which the condition is satisfied. Since in the data the wave length or period was not specified, Arakawa made three sets of calculations by assigning three values of λ, namely, 5 km., 10 km., and 20 km. He concluded that the ray method is valid for the propagation of waves in the deep interior of the earth, but that it fails in the upper part of the crust, especially in the surface layer down to 20 km.

But the actual situation does not seem to be so adverse. The upper layers of the earth's crust have been subject to quite intensive studies by the ray method from the seismograms of near earthquakes. The results are in good agreement with other evidences. It appears therefore that Arakawa's conclusion may be too stringent. This may be due to the fact that the frequencies of the near earthquake waves are usually higher than those of the teleseisms. The effect of the larger velocity gradient in the upper crust is partly compensated by the shorter wave lengths. Furthermore, when an impetus beginning is recorded in the seismogram, the ray method may be applied with a high accuracy. We shall return to this point later.

Another way to look at the matter is with reference to the velocity-depth curve (with the depth as abscissa). At those depths at which the curve is steep the ray method is valid only for short wave lengths, or, preferably, for sharp pulses. For this reason the determination of the fine layering in the earth's interior, especially near the core, must be viewed with care, for the gradient of velocity may not be small there. The so-called discontinuities in the earth's interior should rather be regarded as transition zones. The sharpness of the boundaries of these zones depends greatly on the nature of the seismic disturbance.

Examples of easy calculation may be taken from the case where the velocity is a linear function of depth. In a recent paper Gutenberg[8] gives the following empirical formulae:

For $\bar{P}$, $V = 5.56 + 0.001\,z$ km/sec.
Thickness $\approx$ 18 km.

For P_y, $V = 6.00 + 0.0095\,(z - 18)$ km/sec.
Thickness $\approx$ 22 km.

For P_n, $V = 8.0 + 0.01\,(z - 40)$ km/sec.

[7] H. Arakawa, "The Propagation of Elastic Waves in a Heterogeneous Medium and the Condition for the Validity of the Ray Theory," *Geophys. Mag.*, 7: 155–160 (1933).

[8] B. Gutenberg, "Variations in Physical Properties within the Earth's Crustal Layers," *Am. Jour. Sci.*, Daly vol., 285–312 (1945).

The condition (25) may be applied to each of these waves. For convenience let us write the condition in the form

$$\frac{1}{2\pi}\frac{1}{V}\frac{dV}{dz} << \frac{1}{\lambda} \tag{25d}$$

In order that the depths determined above should have a precise meaning, we assume that they are at least ten times the wave length used. Adopting a wave length of 2 km, we have $1/\lambda = 0.5$. Calculating the left-hand side expression for each of the $\overline{\mathrm{P}}$, P_y, and P_n waves from the foregoing empirical formulae, we find that they are all much smaller than 0.5. Hence, the ray method is really justified in these cases, contrary to the finding of Arakawa.

In seismic prospecting the velocity is of the following order of magnitude:

$$V = 6000 + 0.6z \text{ ft/sec.}$$

The velocity gradient is small and therefore the ray method is generally consistent with the wave theory, provided the thickness of the layer is also large. In fact, the limitation in this case is primarily determined by the nature of the disturbance. If a sharp pulse is used, the method should give reliable result even for a thin bed.

It is probably of interest to consider this condition from another point of view, particularly with respect to seismic prospecting, in which the geometry of the curved ray paths has received considerable attention. When the velocity is a function of the depth only, the radius of curvature ρ of a ray is given by[9]

$$\frac{1}{\rho} = \frac{1}{V}\frac{dV}{dz}\sin i = \frac{\sin i_0}{V_0}\frac{dV}{dz} \tag{26}$$

where i_0 is the angle of incidence to the surface and V_0, the velocity of the body wave near the surface. From this equation it is seen at once that for a linear velocity-depth function, $V = V_0 + az$, ρ is a constant and the paths of the rays are circular arcs in a vertical plane. Substituting (26) in (25), we get

$$\frac{1}{\rho} << \frac{2\pi \sin i}{\lambda} \tag{27}$$

or, for a linear variation,

$$\frac{a}{2\pi V} << \frac{1}{\lambda} \tag{28}$$

where a is the vertical velocity gradient.

[9] B. Gutenberg, "Zur Entwicklung der seismischen Aufschluss-Methoden," *Erg. d. Kosmischen Physik*, 4, 169–218 (1939).

It is perhaps in place to explain here a paradox which is sometimes noted. In exploration work the velocity is of the order of 5,000 feet per second and the measurement of the period is of the order of 0.02 second. These make the wave length about 100 feet. Yet the depth determination is often asserted to be accurate to a few feet, that is, to a small fraction of a wave length. With respect to a continuous wave train this is difficult to visualize. In reality, however, the source of the disturbance is an explosion which generates a sharp pulse. The latter may be conceived as made up in part of a large number of harmonic wave trains of very short wave lengths in accordance with Fourier's theorem. It is the observation of this sharp impulse front which accounts for the high accuracy. In this case the term wave length loses its usual significance.

This situation is also amenable to a mathematical interpretation. If a disturbance is generated in a finite region of a medium, then, at any time, the medium is divided into two regions: a disturbed and an undisturbed. The two are separated by a surface of discontinuity which is the wave front, and its position changes with time. It is known in the theory of differential equations that this surface of discontinuity satisfies the equation of the characteristics of the wave equation. But this latter is no other than the equation of the eikonals which is the governing equation of the ray theory, and the velocity of propagation of this wave front is given by V. The beginning of a seismogram marks the advance of this front. If it is sharp enough, its kinematic relations should conform to the result given by the ray method. But in actual physical phenomena it is quite difficult to visualize an occurrence of a discontinuity in the mathematical sense. What we can observe is usually a concentrated disturbance which may taper off very rapidly but continuously. However, the approximation of this case to a discontinuity is probably a good one.

Determination of the Ray Paths for Some Particular Velocity Functions

The determination of a ray path in seismology is the same as finding the brachistochrone in dynamics. Instead of applying Snell's law, as is usually done, we shall follow Epstein and adopt the method of Jacobi from dynamics. The method is to find a complete integral of the equation of the characteristic function. The partial derivatives of this integral with respect to the nonadditive constants equated to other arbitrary constants will give the parametric equations of the rays. For instance, if a complete integral of (18a) is

$$t = t(x, y, z, \alpha, \beta) + C$$

where α and β are the nonadditive constants and C is the additive one, then

$$\partial t/\partial \alpha = a, \quad \text{and} \quad \partial t/\partial \beta = b$$

are the parametric equations of the rays, a, b being arbitrary constants which may be determined by the conditions of the problem.

Horizontally stratified medium.—In this case the velocity of propagation is a function of the depth z only. The general problem of a complex index of refraction has been discussed by Epstein. For a real V, the problem has attracted considerable attention in the field of applied seismology. To illustrate the method, we shall limit ourselves to this case and work out the explicit solutions for particular velocity functions. We shall assume that

$$V^2 = (V_0 + az)^n \tag{29}$$

which seems to be of sufficient generality for the present purpose. V_0 is the velocity near the ground surface, where z is zero and a is the velocity gradient, both being given constants.

Since the rays must all lie in vertical planes, we may write (18a) in the form

$$\left(\frac{\partial t}{\partial x}\right)^2 + \left(\frac{\partial t}{\partial z}\right)^2 = \frac{1}{(V_0 + az)^n} \tag{30}$$

where the term containing y is omitted. By the method of Lagrange and Charpit we may put

$$\frac{\partial t}{\partial x} = \text{constant} = \frac{1}{k}$$

and obtain the complete integral

$$t = \frac{x}{k} + \int \sqrt{\frac{1}{(V_0 + az)^n} - \frac{1}{k^2}}\, dz \tag{31}$$

The equation of the rays is therefore

$$\frac{\partial t}{\partial k} = -\frac{x}{k^2} + \frac{1}{k^2}\int \frac{(V_0 + az)^{\frac{n}{2}}\, dz}{\sqrt{k^2 - (V_0 + az)^n}} = \text{const.} \tag{32}$$

Making the substitution

$$(V_0 + az)^n = k^2 \sin^2\theta \tag{33}$$

we obtain

$$x + \frac{2k^{\frac{2}{n}}}{na}\int \sin^{\frac{2}{n}}\theta\, d\theta = \text{const.} \tag{34}$$

When $n = 0$, the velocity is a constant. It follows immediately from (32) that the rays are straight lines.

When $n = 2$, the velocity is linear in z. Equation (34) can be integrated at once, and, with (33), we obtain the parametric equations of a family of circles. This solution is well known.

When $n = 1$, the *square* of the velocity is linear in z. The problem is then identical with the classic problem of the Bernoullis solved in 1697. In seismic prospecting, it was solved independently by Ramspeck[10] and by Houston[11] by use of Snell's law. In the present treatment, it is seen that (34), when integrated with $n = 1$, gives

$$x = b - \frac{k^2}{2a}(2\theta - \sin 2\theta) \tag{35}$$

b being a constant. Rewriting (33) in terms of 2θ, we have

$$z = \frac{k^2}{2a}(1 - \cos 2\theta) - V_0/a\,. \tag{36}$$

The last two expressions are the parametric equations of the cycloids. The constants b and k can be determined by the position of the focus and the initial direction of the ray. We can of course integrate (34) by assigning other values of n.

Medium with spherical symmetry.—In this case the velocity of propagation depends on the radial distance r only. This problem has its application in earthquake seismology. Let us assume a velocity function of the form

$$1/V^2 = a^2 r^n \tag{37}$$

a, n, being given constants. In polar coördinates, equation (18a) becomes

$$\left(\frac{\partial t}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial t}{\partial \theta}\right)^2 + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial t}{\partial \varphi}\right)^2 = a^2 r^n \tag{38}$$

By symmetry, t would be independent of the azimuth, and so the last term on the left-hand side must vanish. A complete integral is therefore

$$t = k\theta + \int \sqrt{a^2 r^n - k^2/r^2}\, dr \tag{39}$$

where k is a constant. This integral can be integrated and we obtain

$$t = k\theta + \frac{2k}{n+2}\left[\sqrt{R^2 - 1} - \cos^{-1}(1/R)\right] + C \tag{40}$$

[10] A. Ramspeck, "Der Einfluss eines mit der Tiefe veränderlichen Elastizitätsmoduls auf den Weg elastischer Wellen im Boden," *Zeitschr. f. Geophys.* 15: 148–159 (1939).

[11] C. E. Houston, "Seismic Paths, assuming a Parabolic Increase of Velocity with Depth," *Geophysics*, 4: 242–246 (1939).

where C is an additive constant and R^2 is written for a^2r^{n+2}/k^2. The equation of the rays is obtained from

$$\frac{\partial t}{\partial k} = \theta_0$$

which, from (40), gives

$$r^{\frac{n+2}{2}} = \frac{k}{a} \sec \frac{n+2}{2} (\theta - \theta_0). \tag{41}$$

As before, the constants k and θ_0 can be determined from the position of the focus and the initial direction of the ray.

The solution (42) fails when $n = -2$. But in this case (39) is easily integrated and obtains the relation

$$\log r = \sqrt{a^2/k^2 - 1}\,(\theta - \theta_0) \tag{42}$$

which is a logarithmic spiral.

Medium in which the velocity depends on the distance from a given line.—This case might happen when the sediments are compacted over a buried ridge and might have application in prospecting. If, on the ground surface, a profile is shot in a direction perpendicular to the strike of the ridge, the problem becomes a two-dimensional one. With the z-axis of a cylindrical coördinate system in the direction of the given line, equation (37) with the azimuthal term omitted again applies and solutions (41) and (42) still hold. However, if the profile is shot *not* perpendicular to the direction of the given line, the problem will be a three-dimensional one and in cylindrical coördinates, and we have to begin with the equation

$$\left(\frac{\partial t}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial t}{\partial \theta}\right)^2 + \left(\frac{\partial t}{\partial z}\right)^2 = c^2 r^n \tag{43}$$

If we set

$$\left(\frac{\partial t}{\partial z}\right)^2 = \alpha^2, \qquad \left(\frac{\partial t}{\partial \theta}\right)^2 = \beta^2,$$

where α and β are constants, the complete integral is not easy to evaluate for a general value of n. But the expression becomes simple when $n = -2$. In this case we have[12]

$$t = \alpha z + \beta\theta - E \log \frac{1}{r}(E + F) + F \tag{44}$$

[12] P. G. Tait, "On the Application of Hamilton's Characteristic Function to Special Cases of Constraint," *Trans. Roy. Soc. Edinburgh*, Vol. 24 (1865), or *Sci. Papers*, Vol. 1, pp. 54–73.

where

$$E^2 = c^2 - \beta^2, \qquad F^2 = c^2 - \beta^2 - \alpha^2 r^2$$

The equations of the rays are given by

$$\frac{\partial t}{\partial \alpha} = A, \qquad \frac{\partial t}{\partial \beta} = B$$

which give the parametric equations

$$\alpha(z - A) = \sqrt{c^2 - \beta^2} - \sqrt{c^2 - \beta^2 - \alpha^2 r^2} \tag{45}$$

$$\sqrt{c^2 - \beta^2} = \alpha r \cosh \frac{\sqrt{c^2 - \beta^2}\,(\theta - B)}{\beta} \tag{46}$$

The ray path is therefore the curve of intersection of a sphere and a cylinder.

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